Problem 4.21

The raising and lowering operators change the value of m by one unit:

$$L_+ f_\ell^m = (A_\ell^m)\, f_\ell^{m+1}, \quad L_- f_\ell^m = (B_\ell^m)\, f_\ell^{m-1} \tag{4.120}$$

where A_ℓ^m and B_ℓ^m are constants. *Question*: What *are* they, if the eigenfunctions are to be normalized? *Hint*: First show that $L_\mp$ is the hermitian conjugate of $L_\pm$ (since L_x and L_y are observables, you may assume they are hermitian... but *prove* it if you like); then use Equation 4.112. *Answer*:

$$\begin{aligned} A_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m+1)} = \hbar\sqrt{(\ell-m)(\ell+m+1)}, \\ B_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m-1)} = \hbar\sqrt{(\ell+m)(\ell-m+1)}. \end{aligned} \tag{4.121}$$

Note what happens at the top and bottom of the ladder (i.e. when you apply L_+ to f_ℓ^ℓ or L_- to $f_\ell^{-\ell}$).

Solution

Begin by proving that the components of the (orbital) angular momentum operator, defined classically by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \quad \Rightarrow \quad \begin{cases} L_x = yp_z - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{cases},$$

are hermitian. That is, the aim is to show that

$$\langle f \,|\, L_j \,|\, g \rangle = \langle f \,|\, L_j^\dagger \,|\, g \rangle, \quad j = 1, 2, 3,$$

where f and g are complex-valued functions that tend to zero as $|\mathbf{x}| \to \infty$. Suppose that $f = u + iv$, where u and v are real functions.

$$\begin{aligned} \langle f \,|\, L_j \,|\, g \rangle &= \iiint\limits_{\text{all space}} f^* L_j g \, d\mathcal{V} \\ &= \iiint\limits_{\text{all space}} f^* \left(\sum_{k=1}^3 \sum_{l=1}^3 \varepsilon_{jkl} x_k p_l \right) g \, d\mathcal{V} \\ &= \sum_{k=1}^3 \sum_{l=1}^3 \varepsilon_{jkl} \iiint\limits_{\text{all space}} f^* x_k p_l g \, d\mathcal{V} \\ &= \sum_{k=1}^3 \sum_{l=1}^3 \varepsilon_{jkl} \iiint\limits_{\text{all space}} f^* x_k \left(-i\hbar \frac{\partial}{\partial x_l} \right) g \, d\mathcal{V} \end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\langle f \,|\, L_j \,|\, g\rangle &= -i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} f^* x_k \frac{\partial g}{\partial x_l}\, d\mathcal{V} \\
&= -i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^* x_k \frac{\partial g}{\partial x_l}\, dx_j\, dx_k\, dx_l \\
&= -i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \int_{-\infty}^{\infty} x_k \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f^* \frac{\partial g}{\partial x_l}\, dx_l\right) dx_j\, dx_k \\
&= -i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \int_{-\infty}^{\infty} x_k \int_{-\infty}^{\infty} \Bigg(\underbrace{f^* g\Big|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \frac{\partial f^*}{\partial x_l} g\, dx_l\Bigg) dx_j\, dx_k \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_k \frac{\partial f^*}{\partial x_l} g\, dx_j\, dx_k\, dx_l \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} x_k \frac{\partial f^*}{\partial x_l} g\, d\mathcal{V} \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} x_k \left[\frac{\partial}{\partial x_l}(u - iv)\right] g\, d\mathcal{V} \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} x_k \left(\frac{\partial u}{\partial x_l} - i\frac{\partial v}{\partial x_l}\right) g\, d\mathcal{V} \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} x_k \left(\frac{\partial u}{\partial x_l} + i\frac{\partial v}{\partial x_l}\right)^* g\, d\mathcal{V} \\
&= i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \iiint\limits_{\text{all space}} x_k \left(\frac{\partial f}{\partial x_l}\right)^* g\, d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \left(-i\hbar \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} x_k \frac{\partial f}{\partial x_l}\right)^* g\, d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} x_k \left(-i\hbar \frac{\partial}{\partial x_l}\right) f\right]^* g\, d\mathcal{V}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle f \,|\, L_j \,|\, g\rangle &= \iiint\limits_{\text{all space}} \left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} x_k p_l\right) f\right]^* g\, d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} (L_j f)^* g\, d\mathcal{V} \\
&= \langle f \,|\, L_j^\dagger \,|\, g\rangle,
\end{aligned}$$

which means the components of the orbital angular momentum operator are hermitian. L_- and L_+ are the ladder operators associated with the orbital angular momentum eigenstates f_ℓ^m. They're defined as

$$\begin{cases} L_- = L_x - iL_y \\ L_+ = L_x + iL_y \end{cases}.$$

Show that L_- is the hermitian conjugate of L_+: $L_- = L_+^\dagger$.

$$\begin{aligned}
\langle f \,|\, L_- \,|\, g\rangle &= \langle f \,|\, L_x - iL_y \,|\, g\rangle \\
&= \langle f \,|\, L_x \,|\, g\rangle - i\langle f \,|\, L_y \,|\, g\rangle \\
&= \langle f \,|\, L_x^\dagger \,|\, g\rangle - i\langle f \,|\, L_y^\dagger \,|\, g\rangle \\
&= \langle f \,|\, L_x^\dagger - iL_y^\dagger \,|\, g\rangle \\
&= \langle f \,|\, (L_x + iL_y)^\dagger \,|\, g\rangle \\
&= \langle f \,|\, L_+^\dagger \,|\, g\rangle
\end{aligned}$$

Show that L_+ is the hermitian conjugate of L_-: $L_+ = L_-^\dagger$.

$$\begin{aligned}
\langle f \,|\, L_+ \,|\, g\rangle &= \langle f \,|\, L_x + iL_y \,|\, g\rangle \\
&= \langle f \,|\, L_x \,|\, g\rangle + i\langle f \,|\, L_y \,|\, g\rangle \\
&= \langle f \,|\, L_x^\dagger \,|\, g\rangle + i\langle f \,|\, L_y^\dagger \,|\, g\rangle \\
&= \langle f \,|\, L_x^\dagger + iL_y^\dagger \,|\, g\rangle \\
&= \langle f \,|\, (L_x - iL_y)^\dagger \,|\, g\rangle \\
&= \langle f \,|\, L_-^\dagger \,|\, g\rangle
\end{aligned}$$

The square of the total angular momentum is $L^2 = L_x^2 + L_y^2 + L_z^2$. The eigenvalue problems for it and for L_z in Equation 4.118 on page 160 are necessary to determine A_ℓ^m and B_ℓ^m.

$$\begin{cases} L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m & \ell = 0, 1/2, 1, 3/2, \ldots \\ L_z f_\ell^m = \hbar m f_\ell^m & m = -\ell, -\ell+1, \ldots, \ell-1, \ell \end{cases} \tag{4.118}$$

To find A_ℓ^m, consider the following. Note $[L_x, L_y] = i\hbar L_z$.

$$\langle f_\ell^m \,|\, L_- L_+ \,|\, f_\ell^m \rangle = \iiint\limits_{\text{all space}} (f_\ell^m)^* L_- L_+ f_\ell^m \, d\mathcal{V}$$

$$\langle f_\ell^m \,|\, L_+^\dagger L_+ \,|\, f_\ell^m \rangle = \iiint\limits_{\text{all space}} (f_\ell^m)^* (L_x - iL_y)(L_x + iL_y) f_\ell^m \, d\mathcal{V}$$

$$\left(\langle f_\ell^m | L_+^\dagger\right)(L_+ | f_\ell^m\rangle) = \iiint\limits_{\text{all space}} (f_\ell^m)^* (L_x - iL_y)(L_x f_\ell^m + iL_y f_\ell^m) \, d\mathcal{V}$$

$$(L_+ | f_\ell^m\rangle)^* (L_+ | f_\ell^m\rangle) = \iiint\limits_{\text{all space}} (f_\ell^m)^* \left[L_x(L_x f_\ell^m + iL_y f_\ell^m) - iL_y(L_x f_\ell^m + iL_y f_\ell^m)\right] d\mathcal{V}$$

$$\left(A_\ell^m | f_\ell^{m+1}\rangle\right)^* \left(A_\ell^m | f_\ell^{m+1}\rangle\right) = \iiint\limits_{\text{all space}} (f_\ell^m)^* \left[L_x^2 + L_y^2 + i(L_x L_y - L_y L_x)\right] f_\ell^m \, d\mathcal{V}$$

$$\langle f_\ell^{m+1} \,|\, (A_\ell^m)^* A_\ell^m \,|\, f_\ell^{m+1} \rangle = \iiint\limits_{\text{all space}} (f_\ell^m)^* \left[(L^2 - L_z^2) + i[L_x, L_y]\right] f_\ell^m \, d\mathcal{V}$$

$$(A_\ell^m)^* A_\ell^m \underbrace{\langle f_\ell^{m+1} \,|\, f_\ell^{m+1} \rangle}_{=1} = \iiint\limits_{\text{all space}} (f_\ell^m)^* (L^2 - L_z^2 - \hbar L_z) f_\ell^m \, d\mathcal{V}$$

$$(A_\ell^m)^* A_\ell^m = \iiint\limits_{\text{all space}} (f_\ell^m)^* (L^2 f_\ell^m - L_z^2 f_\ell^m - \hbar L_z f_\ell^m) \, d\mathcal{V}$$

$$|A_\ell^m|^2 = \iiint\limits_{\text{all space}} (f_\ell^m)^* \left[\hbar^2 \ell(\ell+1) f_\ell^m - (\hbar m)^2 f_\ell^m - \hbar(\hbar m) f_\ell^m\right] d\mathcal{V}$$

$$= \iiint\limits_{\text{all space}} (f_\ell^m)^* \hbar^2 [\ell(\ell+1) - m(m+1)] f_\ell^m \, d\mathcal{V}$$

$$= \hbar^2 [\ell(\ell+1) - m(m+1)] \underbrace{\iiint\limits_{\text{all space}} (f_\ell^m)^* f_\ell^m \, d\mathcal{V}}_{=1}$$

Therefore, taking the square root of both sides,

$$\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}.}$$

To find B_ℓ^m, consider the following. Note $[L_x, L_y] = i\hbar L_z$.

$$\langle f_\ell^m \mid L_+ L_- \mid f_\ell^m \rangle = \iiint_{\text{all space}} (f_\ell^m)^* L_+ L_- f_\ell^m \, d\mathcal{V}$$

$$\langle f_\ell^m \mid L_-^\dagger L_- \mid f_\ell^m \rangle = \iiint_{\text{all space}} (f_\ell^m)^* (L_x + iL_y)(L_x - iL_y) f_\ell^m \, d\mathcal{V}$$

$$\left(\langle f_\ell^m | L_-^\dagger\right)(L_- | f_\ell^m \rangle) = \iiint_{\text{all space}} (f_\ell^m)^* (L_x + iL_y)(L_x f_\ell^m - iL_y f_\ell^m) \, d\mathcal{V}$$

$$(L_- | f_\ell^m \rangle)^* (L_- | f_\ell^m \rangle) = \iiint_{\text{all space}} (f_\ell^m)^* \left[L_x(L_x f_\ell^m - iL_y f_\ell^m) + iL_y(L_x f_\ell^m - iL_y f_\ell^m)\right] d\mathcal{V}$$

$$\left(B_\ell^m | f_\ell^{m-1} \rangle\right)^* \left(B_\ell^m | f_\ell^{m-1} \rangle\right) = \iiint_{\text{all space}} (f_\ell^m)^* \left[L_x^2 + L_y^2 - i(L_x L_y - L_y L_x)\right] f_\ell^m \, d\mathcal{V}$$

$$\langle f_\ell^{m-1} \mid (B_\ell^m)^* B_\ell^m \mid f_\ell^{m-1} \rangle = \iiint_{\text{all space}} (f_\ell^m)^* \left[(L^2 - L_z^2) - i[L_x, L_y]\right] f_\ell^m \, d\mathcal{V}$$

$$(B_\ell^m)^* B_\ell^m \underbrace{\langle f_\ell^{m-1} \mid f_\ell^{m-1} \rangle}_{=1} = \iiint_{\text{all space}} (f_\ell^m)^* (L^2 - L_z^2 + \hbar L_z) f_\ell^m \, d\mathcal{V}$$

$$(B_\ell^m)^* B_\ell^m = \iiint_{\text{all space}} (f_\ell^m)^* (L^2 f_\ell^m - L_z^2 f_\ell^m + \hbar L_z f_\ell^m) \, d\mathcal{V}$$

$$|B_\ell^m|^2 = \iiint_{\text{all space}} (f_\ell^m)^* \left[\hbar^2 \ell(\ell+1) f_\ell^m - (\hbar m)^2 f_\ell^m + \hbar(\hbar m) f_\ell^m\right] d\mathcal{V}$$

$$= \iiint_{\text{all space}} (f_\ell^m)^* \hbar^2 [\ell(\ell+1) - m(m-1)] f_\ell^m \, d\mathcal{V}$$

$$= \hbar^2 [\ell(\ell+1) - m(m-1)] \underbrace{\iiint_{\text{all space}} (f_\ell^m)^* f_\ell^m \, d\mathcal{V}}_{=1}$$

Therefore, taking the square root of both sides,

$$\boxed{B_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m-1)}.}$$

Observe that applying the raising operator to the top rung ($m = \ell$) yields zero.

$$\begin{aligned} L_+ f_\ell^\ell &= A_\ell^\ell f_\ell^{\ell+1} \\ &= \hbar\sqrt{\ell(\ell+1) - \ell(\ell+1)} f_\ell^{\ell+1} \\ &= 0 \end{aligned}$$

Observe that applying the lowering operator to the bottom rung ($m = -\ell$) also yields zero.

$$\begin{aligned} L_- f_\ell^{-\ell} &= B_\ell^{-\ell} f_\ell^{-\ell-1} \\ &= \hbar\sqrt{\ell(\ell+1) - (-\ell)[(-\ell) - 1]} f_\ell^{-\ell-1} \\ &= \hbar\sqrt{\ell(\ell+1) - \ell(\ell+1)} f_\ell^{-\ell-1} \\ &= 0 \end{aligned}$$